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THE GENESIS OF LAMÉ'S EQUATION.

BY

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THE GENESIS OF LAMÉ'S EQUATION.

By Prof. E. H. NEVILLE.

THIS note was provoked by the assertion in the chapter added to *Modern Analysis* in the third edition, that "all the analytical proofs" of Lamé's expression for Laplace's equation in terms of orthogonal curvilinear coordinates "are extremely long and cumbrous", and by the restriction to real variables in the treatment of confocal coordinates by Whittaker and Watson.

The note begins therefore with an analytical transformation of Laplace's operator, which is quite short, although it proceeds from first principles, and care is taken to explain the meaning of the steps if the variables are complex.

It is then pointed out that the use of confocal coordinates does not presuppose the variables to be real, and the familiar formulæ are found without reference to geometry. An account of the origin of ellipsoidal harmonics and of the relation of the different species to Lamé's equation follows naturally, and the note ends with a transformation into confocal coordinates of the operator introduced by Niven.

There is no novelty in the results, and the methods are straightforward; all that is claimed is that the algebra is brief and to the point.

1. If real variables x, y, z, the coordinates of a point in space referred to rectangular axes, are functions of three independent variables t_1, t_2, t_3, the direction ratios γ^{r1}, γ^{r2}, γ^{r3} of a directed tangent to the curve along which t_r varies and the other two variables are constant have the values dx/ds_r, dy/ds_r, dz/ds_r, where s_r is the arc of the directed curve measured from some fixed point upon it. If ds_r/dt_r is denoted by $H_{(r)}$, the rate of change of any function ϕ along the curve is $\phi_r/H_{(r)}$, where ϕ_r denotes $\partial\phi/\partial t_r$, the function ϕ being supposed expressed as a function of t_1, t_2, t_3; in particular,

1.1 $$\gamma^{r1} = x_r/H_{(r)},\quad \gamma^{r2} = y_r/H_{(r)},\quad \gamma^{r3} = z_r/H_{(r)},$$

and since

1.2 $$\sum_s (\gamma^{rs})^2 = 1,$$

it follows that

1.3 $$H_{(r)}^2 = Su_r^2,$$

where with Lamé we write[*] $SF(u)$ to denote $F(x)+F(y)+F(z)$. The transformation from x, y, z to t_1, t_2, t_3 is triply orthogonal if

1.4 $$\sum_s \gamma^{qs}\gamma^{rs} = 0, \quad q \neq r,$$

and therefore if

1.5 $$Su_q u_r = 0, \quad q \neq r.$$

The tangents to the curves of reference then form a direct system if the determinant (γ^{rs}), whose square necessarily has the value 1, itself has the value 1, not -1. If the determinant is equal to -1, an interchange of t_2 and t_3, altering only the notation of the transformation, gives a determinant with the value 1, and the same effect is produced by changing the sign of one of the three functions $H_{(1)}$, $H_{(2)}$, $H_{(3)}$; thus without any loss of generality it is possible to assume for a triply orthogonal system

1.6 $$(\gamma^{rs}) = 1,$$

that is,

1.7 $$\partial(x, y, z)/\partial(t_1, t_2, t_3) = H_{(1)}H_{(2)}H_{(3)}.$$

If a vector has components ξ, η, ζ relative to the axes of coordinates and τ_1, τ_2, τ_3 relative to the frame composed of the tangents to the curves of reference, then on the one hand

1.8 $$\xi = \sum_r \gamma^{r1}\tau_r, \quad \eta = \sum_r \gamma^{r2}\tau_r, \quad \zeta = \sum_r \gamma^{r3}\tau_r,$$

and on the other hand

1.9 $$\tau_r = \gamma^{r1}\xi + \gamma^{r2}\eta + \gamma^{r3}\zeta, \quad r = 1, 2, 3;$$

since the vector is arbitrary, each of the sets of formulæ 1.8, 1.9 is *equivalent* to the other.

2. When a transformation is made from a set of three complex variables x, y, z to another set t_1, t_2, t_3, all the formulæ

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[*] *Not* $SF(x)$; the superiority of Lamé's notation to that commonly ascribed to him is obvious in such a case as $x\delta^2\phi / \delta x^2 + y\delta^2\phi / \delta x\delta y + z\delta^2\phi / \delta x\delta z$, which becomes simply $Su\delta^2\phi / \delta x\delta u$.

of section 1 remain true, but some of them express *definitions.* By 1.3 are defined the squares of three arc functions; in this paper we suffer no loss of generality by supposing that not one of these squares is identically zero, for three distinct directions cannot be mutually perpendicular if one of them is nul; we can therefore suppose also that the range of the variable is so restricted as to render the arc functions themselves single valued. The arc of the r^{th} "curve" of reference between two given points is definable as $\int H_{(r)}dt_r$, but the curve is a two-dimensional region. The functions defined by 1.1 are called direction ratios, and they satisfy 1.2, which is now a consequence of 1.3, not its basis.

The transformation from x, y, z to t_1, t_2, t_3 is *called* triply orthogonal if 1.5 is true, that is, if 1.4 holds, and the transformation is said to be direct if in addition 1.6 and 1.7 are satisfied; there is no loss of generality in supposing the transformation to be direct. To know that 1.8 and 1.9 are still equivalent relations between any two sets of complex numbers ξ, η, ζ and τ_1, τ_2, τ_3, it is not necessary to have a theory of vectors in complex space; it is an immediate algebraic corollary of 1.2 and 1.4 that 1.9 follows from 1.8, and 1.6 is sufficient therefore to secure that 1.8 follows from 1.9.

From the set of formulæ 1.1, written as

$$x_r = \gamma^{r1} H_{(r)}, \quad y_r = \gamma^{r2} H_{(r)}, \quad z_r = \gamma^{r3} H_{(r)},$$

combined with the equalities

$$x_{rs} = x_{sr}, \quad y_{rs} = y_{sr}, \quad z_{rs} = z_{sr},$$

we have a set of important relations comprised in the one typical formula

2.1 $$\gamma_s^{rt} H_{(r)} + \gamma^{rt} H_{(r)s} = \gamma_r^{st} H_{(s)} + \gamma^{st} H_{(s)r},$$

where $H_{(r)s}$, $H_{(s)r}$ denote $\partial H_{(r)}/\partial t_s$, $\partial H_{(s)}/\partial t_r$. From 1.2, with s, t for r, s,

2.2 $$\sum_t \gamma^{st} \gamma_q^{st} = 0,$$

and therefore from 2.1

2.3 $$\left(\sum_t \gamma^{st} \gamma_s^{rt}\right) H_{(r)} = H_{(s)r}, \quad r \neq s.$$

3. For any function ϕ, expressed on the one hand as a function of x, y, z, and on the other as a function of t_1, t_2, t_3,

we have

3.1 $$\phi_r = \phi_x x_r + \phi_y y_r + \phi_z z_r, \quad r = 1, 2, 3,$$

and therefore, from 1.1,

3.2 $$\phi_r / H_{(r)} = \gamma^{r1}\phi_x + \gamma^{r2}\phi_y + \gamma^{r3}\phi_z, \quad r = 1, 2, 3.$$

This is a set of relations of the form 1.9, and is therefore equivalent to

3.3 $$\phi_x = \sum_r \gamma^{r1}\phi_r / H_{(r)}, \quad \phi_y = \sum_r \gamma^{r2}\phi_r / H_{(r)}, \quad \phi_z = \sum_r \gamma^{r3}\phi_r / H_{(r)}.$$

If ψ denotes ϕ_x, a double application of 3.3 gives

$$\phi_{xx} = \psi_x = \sum_s \gamma^{s1}\psi_s / H_{(s)} = \sum_s (\gamma^{s1} / H_{(s)})\,\partial\Big(\sum_r \gamma^{r1}\phi_r / H_{(r)}\Big) / \partial t_s,$$

that is

3.4 $$\phi_{xx} = \sum_r \Big\{\Big(\sum_s \gamma^{s1}\gamma^{r1}/H_{(s)}\Big)\,\partial(\phi_r/H_{(r)})/\partial t_s + \Big(\sum_s \gamma^{s1}\gamma_s^{r1}/H_{(s)}\Big)\phi_r/H_{(r)}\Big\},$$

and by addition from the three formulæ of this type we have at once, utilising 1.2, 1.4, 2.2 with s for q, and 2.3,

$$\nabla^2\phi \equiv \phi_{xx} + \phi_{yy} + \phi_{zz}$$

$$= \sum_r \left\{ \frac{1}{H_{(r)}} \frac{\partial}{\partial t_r} \left(\frac{\phi_r}{H_{(r)}} \right) + \sum_{s \neq r} \left(\frac{H_{(s)r}}{H_{(r)}H_{(s)}} \, \frac{\phi_r}{H_{(r)}} \right) \right\};$$

since identically

$$\sum_{s \neq r} \frac{1}{H_{(s)}} \frac{\partial H_{(s)}}{\partial t_r} = \frac{1}{\Pi_{(r)}'} \frac{\partial \Pi_{(r)}'}{\partial t_r},$$

where $\Pi_{(r)}'$ denotes $H_{(1)}H_{(2)}H_{(3)}/H_{(r)}$, the formula is equivalent to

$$\nabla^2\phi = \sum_r \frac{1}{H_{(r)}\Pi_{(r)}'} \left\{ \Pi_{(r)}' \frac{\partial}{\partial t_r} \left(\frac{\phi_r}{H_{(r)}} \right) + \frac{\phi_r}{H_{(r)}} \frac{\partial \Pi_{(r)}'}{\partial t_r} \right\},$$

that is, to

3.6 $$\nabla^2\phi = \frac{1}{H_{(1)}H_{(2)}H_{(3)}} \sum_r \frac{\partial}{\partial t_r} \left(\frac{\Pi_{(r)}'}{H_{(r)}} \frac{\partial \phi}{\partial t_r} \right),$$

the familiar result discovered by Lamé.

4. Let a, b, c be three fixed finite complex numbers, subject only to the condition that no two of them are equal. Then if x, y, z are complex variables, the equation

4.1 $$x^2/(a+\theta) + y^2/(b+\theta) + z^2/(c+\theta) = 1,$$

regarded as an equation in a variable θ, has three roots. If x, y, z are unrestricted the roots have to be regarded as the three branches of a single function, but if the joint variation of x, y, z is adequately limited, the three roots are separate single-valued functions of x, y, z. These roots, which will be denoted by θ_1, θ_2, θ_3, form the set of confocal variables associated with x, y, z by the base numbers a, b, c.

We shall write a_θ, b_θ, c_θ for $a+\theta$, $b+\theta$, $c+\theta$ and a_p, b_p, c_p for $a+\theta_p$, $b+\theta_p$, $c+\theta_p$; obviously

$$4.2 \qquad a_p - a_q = b_p - b_q = c_p - c_q = \theta_p - \theta_q.$$

In using Lamé's symbol of summation, we shall write $SF(d, u)$ for $F(a, x) + F(b, y) + F(c, z)$. Thus the equation 4.1 can be written

$$Su^2/(d+\theta) = 1.$$

For any value of θ the function

$$Su^2/(d+\theta) - 1$$

will be denoted by $\Theta(\theta)$.

The expression of x^2, y^2, z^2 as explicit functions of the confocal variables is immediate, for $(a+\theta)(b+\theta)(c+\theta)\,\Theta(\theta)$ is a cubic polynomial in θ in which the coefficient of θ^3 is -1, and the only function of this kind with θ_1, θ_2, θ_3 for roots is $(\theta_1-\theta)(\theta_2-\theta)(\theta_3-\theta)$. Hence identically

$$(4.3) \qquad \Theta(\theta) = \frac{(\theta_1-\theta)(\theta_2-\theta)(\theta_3-\theta)}{(a+\theta)(b+\theta)(c+\theta)},$$

and decomposing the fraction on the right we have

$$S\,\frac{u^2}{d+\theta} \equiv \frac{a_1a_2a_3}{(a-b)(a-c)}\cdot\frac{1}{a+\theta} + \frac{b_1b_2b_3}{(b-c)(b-a)}\cdot\frac{1}{b+\theta} + \frac{c_1c_2c_3}{(c-a)(c-b)}\cdot\frac{1}{c+\theta};$$

and therefore

$$4.4 \quad x^2 = \frac{a_1a_2a_3}{(a-b)(a-c)},\ y^2 = \frac{b_1b_2b_3}{(b-c)(b-a)},\ z^2 = \frac{c_1c_2c_3}{(c-a)(c-b)}.$$

While x^2, y^2, z^2 are seen to be single-valued functions of θ_1, θ_2, θ_3, the variables x, y, z are single-valued only if restrictions are imposed on their variation.

It is assumed that in the plane associated with the complex variable θ_p there are cuts from the three points $-a$, $-b$, $-c$ to infinity; on the cut plane the square roots of the functions

a_p, b_p, c_p are severally single-valued, and with a definite choice among these square roots and among the square roots of the constants $b-c$, $c-a$, $a-b$, the functions x, y, z become uniform within their restricted domains.

It is convenient to denote the product of selected square roots of a_p, b_p, c_p by Δ_p; thus

$$\Delta_p^2 = a_p b_p c_p, \quad p = 1, 2, 3,$$

and Δ_q is in fact uniform through the range considered.

Differentiating logarithmically the formulæ* in 4.4, we have

4.5 $$x_p = x/2a_p, \quad y_p = y/2b_p, \quad z_p = z/2c_p,$$

and therefore with the notation of 1.3

4.6 $$H_{(p)}^2 = Su_p^2 = \tfrac{1}{4} Su^2/d_p^2.$$

If x, y, z and θ_1, θ_2, θ_3 are regarded as constants,

$$d\{\Theta(\theta)\}/d\theta = -Su^2/d_\theta^2;$$

hence $$4H_{(p)}^2 = -[d\Theta(\theta)/d\theta]_{\theta=\theta_p}$$

$$= -\lim_{\theta\to\theta_p} \Theta(\theta)/(\theta-\theta_p)$$

$$= \prod_q{}' (\theta_q - \theta_p)/\Delta_p^2,$$

from 4.3, where the product is for the two values of q that are different from p. That is to say, $H_{(1)}^2$, $H_{(2)}^2$, $H_{(3)}^2$ are given in terms of the confocal variables alone by

4.7 $$4\Delta_1^2 H_{(1)}^2 = (\theta_2 - \theta_1)(\theta_3 - \theta_1),$$

$$4\Delta_2^2 H_{(2)}^2 = (\theta_3 - \theta_2)(\theta_1 - \theta_2),$$

$$4\Delta_3^2 H_{(3)}^2 = (\theta_1 - \theta_3)(\theta_2 - \theta_3).$$

From 4.2 we have

$$Su^2/d_p - Su^2/d_q \equiv -(\theta_p - \theta_q)\, Su^2/d_p d_q,$$

and therefore since each of the sums on the left has the value unity,

4.8 $$Su^2/d_p d_q = 0, \quad p \neq q,$$

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* The "geometrical" argument, from the normal to the surface of reference, is more illuminating, but I am deliberately avoiding any use of complex space.

whence from 4.5

4.9 $$Su_p u_q = 0, \quad p \neq q:$$

the transformation to confocal variables is triply orthogonal, whatever the values of a, b, c.

5. Since $H_{(1)}$, $H_{(2)}$, $H_{(3)}$ are given by 4.7 as functions of θ_1, θ_2, θ_3, the expression of Laplace's operator in terms of confocal variables is deducible from 3.6; since Δ_p is a function of θ_p only, the formulæ in 4.7 imply that $\Pi_{(r)}'/\Delta_r H_{(r)}$ does not depend on θ_r, whence

5.1 $$\frac{\partial}{\partial\theta_r}\left(\frac{\Pi_{(r)}'}{H_{(r)}}\frac{\partial\phi}{\partial\theta_r}\right) = \frac{\Pi_{(r)}'}{\Delta_r H_{(r)}}\frac{\partial}{\partial\theta_r}\left(\Delta_r\frac{\partial\phi}{\partial\theta_r}\right)$$
$$= \frac{4H_{(1)}H_{(2)}H_{(3)}\Delta_r}{4\Delta_r^2 H_{(r)}^2}\frac{\partial}{\partial\theta_r}\left(\Delta_r\frac{\partial\phi}{\partial\theta_r}\right),$$

and therefore

5.2 $$\tfrac{1}{4}(\theta_2-\theta_3)(\theta_3-\theta_1)(\theta_1-\theta_2)\nabla^2\phi = (\theta_2-\theta_3)\,\Delta_1\frac{\partial}{\partial\theta_1}\left(\Delta_1\frac{\partial\phi}{\partial\theta_1}\right)$$
$$+(\theta_3-\theta_1)\,\Delta_2\frac{\partial}{\partial\theta_2}\left(\Delta_2\frac{\partial\phi}{\partial\theta_2}\right)+(\theta_1-\theta_2)\,\Delta_3\frac{\partial}{\partial\theta_3}\left(\Delta_3\frac{\partial\phi}{\partial\theta_3}\right).$$

The irrationality is only apparent, since

5.3 $$\Delta_r\frac{\partial}{\partial\theta_r}\left(\Delta_r\frac{\partial\phi}{\partial\theta_r}\right) = \Delta_r^2\frac{\partial^2\phi}{\partial\theta_r^2}+\tfrac{1}{2}\frac{\partial\Delta_r^2}{\partial\theta_r}\frac{\partial\phi}{\partial\theta_r},$$

or explicitly

5.4 $$\Delta_r\frac{\partial}{\partial\theta_r}\left(\Delta_r\frac{\partial\phi}{\partial\theta_r}\right) = a_r b_r c_r\left\{\phi_{rr}+\tfrac{1}{2}\left(\frac{1}{a_r}+\frac{1}{b_r}+\frac{1}{c_r}\right)\phi_r\right\}.$$

If ϕ is a function such that

5.5 $$\Delta_1\partial(\Delta_1\phi_1)/\partial\theta_1 : \Delta_2\partial(\Delta_2\phi_2)/\partial\theta_2 : \Delta_3\partial(\Delta_3\phi_3)/\partial\theta_3 = \Psi_{(1)}:\Psi_{(2)}:\Psi_{(3)},$$

where $\Psi_{(r)}$ depends only on $\theta_{(r)}$, the equation $\nabla^2\phi = 0$ is equivalent to

5.6 $$(\theta_2-\theta_3)\,\Psi_{(1)}+(\theta_3-\theta_1)\,\Psi_{(2)}+(\theta_1-\theta_2)\,\Psi_{(3)} = 0,$$

that is, to

$$\begin{vmatrix} \Psi_{(1)} & \theta_1 & 1 \\ \Psi_{(2)} & \theta_2 & 1 \\ \Psi_{(3)} & \theta_3 & 1 \end{vmatrix} = 0,$$

and therefore there are multipliers A, B such that

$$5.7 \qquad \Psi_{(1)} = A\theta_1 + B, \quad \Psi_{(2)} = A\theta_2 + B, \quad \Psi_{(3)} = A\theta_3 + B;$$

from the last two equations

$$A = (\Psi_{(2)} - \Psi_{(3)}) / (\theta_2 - \theta_3), \quad B = (\theta_2\Psi_{(3)} - \theta_3\Psi_{(2)}) / (\theta_2 - \theta_3),$$

showing that A and B do not involve θ_1: similarly A and B are independent of θ_2 and θ_3, and therefore these multipliers are absolute constants.

6. A *normal* solution of Laplace's equation in confocal variables is a solution of the form $\Lambda(\theta_1)\,\mathrm{M}(\theta_2)\,\mathrm{N}(\theta_3)$, the product of three functions each of a single variable. Since x^2, y^2, z^2 are themselves products of this kind, there is no greater generality in dealing with a function $x^l y^m z^n \Lambda\mathrm{MN}$ than in considering $\Lambda\mathrm{MN}$, but it is useful to have the constants l, m, n at our disposal. Writing

$$6.1 \qquad \phi = x^l y^m z^n \Lambda\mathrm{MN}$$

we have

$$6.2 \qquad \frac{\phi_r}{\phi} = \frac{K_r'}{K_r} + \frac{lx_r}{x} + \frac{my_r}{y} + \frac{nz_r}{z},$$

where K_1, K_2, K_3 denote $\Lambda(\theta_1)$, $\mathrm{M}(\theta_2)$, $\mathrm{N}(\theta_3)$, and primes indicate differentiation with respect to the variable involved. Substituting from 4.5,

$$6.3 \qquad \frac{\phi_r}{\phi} = \frac{K_r'}{K_r} + \tfrac{1}{2}\left(\frac{l}{a_r} + \frac{m}{b_r} + \frac{n}{c_r}\right),$$

and therefore

$$6.4 \qquad \frac{\phi_{rr}}{\phi} - \left(\frac{\phi_r}{\phi}\right)^2 = \frac{K_r''}{K_r} - \left(\frac{K_r'}{K_r}\right)^2 - \tfrac{1}{2}\left(\frac{l}{a_r^2} + \frac{m}{b_r^2} + \frac{n}{c_r^2}\right);$$

since from 6.3

$$\left(\frac{\phi_r}{\phi}\right)^2 - \left(\frac{K_r'}{K_r}\right)^2 = \tfrac{1}{2}\left(\frac{l}{a_r} + \frac{m}{b_r} + \frac{n}{c_r}\right)\left\{\frac{2K_r'}{K_r} + \tfrac{1}{2}\left(\frac{l}{a_r} + \frac{m}{b_r} + \frac{n}{c_r}\right)\right\},$$

5.4 and 6.4 imply

$$\begin{aligned}\frac{\Delta_r}{\phi}\frac{\partial}{\partial\theta_r}\left(\Delta_r\frac{\partial\phi}{\partial\theta_r}\right) = a_r b_r c_r \Bigg[&\frac{K_r''}{K_r} + \left(\frac{l}{a_r} + \frac{m}{b_r} + \frac{n}{c_r}\right)\frac{K_r'}{K_r} \\ &+ \tfrac{1}{4}\left(\frac{l}{a_r} + \frac{m}{b_r} + \frac{n}{c_r}\right)^2 - \tfrac{1}{2}\left(\frac{l}{a_r^2} + \frac{m}{b_r^2} + \frac{n}{c_r^2}\right) \\ &+ \tfrac{1}{2}\left(\frac{1}{a_r} + \frac{1}{b_r} + \frac{1}{c_r}\right)\left\{\frac{K_r'}{K_r} + \tfrac{1}{2}\left(\frac{l}{a_r} + \frac{m}{b_r} + \frac{n}{c_r}\right)\right\}\Bigg],\end{aligned}$$

that is

$$6.5 \quad \frac{\Delta_r}{\phi}\frac{\partial}{\partial\theta_r}\left(\Delta_r\frac{\partial\phi}{\partial\theta_r}\right)=\frac{a_rb_rc_r}{K_r}\left[K_r''+\left(\frac{l+\frac{1}{2}}{a_r}+\frac{m+\frac{1}{2}}{b_r}+\frac{n+\frac{1}{2}}{c_r}\right)K_r\right.$$

$$\left.+\tfrac{1}{4}\left\{\left(\frac{l}{a_r}+\frac{m}{b_r}+\frac{n}{c_r}\right)\left(\frac{l+1}{a_r}+\frac{m+1}{b_r}+\frac{n+1}{c_r}\right)-2\left(\frac{l}{a_r^{\,2}}+\frac{m}{b_r^{\,2}}+\frac{n}{c_r^{\,2}}\right)\right\}K_r\right].$$

It follows that the function $x^ly^mz^n\,\Lambda(\theta_1)\,\mathrm{M}(\theta_2)\,\mathrm{N}(\theta_3)$ is a solution of Laplace's equation if and only if the functions $\Lambda(\theta)$, $\mathrm{M}(\theta)$, $\mathrm{N}(\theta)$ are three solutions of the differential equation

$$6.6 \quad \frac{d^2K}{d\theta^2}+\left(\frac{l+\frac{1}{2}}{a+\theta}+\frac{m+\frac{1}{2}}{b+\theta}+\frac{n+\frac{1}{2}}{c+\theta}\right)\frac{dK}{d\theta}$$

$$+\tfrac{1}{4}\left[\left\{\frac{l}{a+\theta}+\frac{m}{b+\theta}+\frac{n}{c+\theta}\right\}\left\{\frac{l+1}{a+\theta}+\frac{m+1}{b+\theta}+\frac{n+1}{c+\theta}\right\}\right.$$

$$\left.-2\left\{\frac{l}{(a+\theta)^2}+\frac{m}{(b+\theta)^2}+\frac{n}{(c+\theta)^2}\right\}-\frac{A\theta+B}{(a+\theta)(b+\theta)(c+\theta)}\right]K=0,$$

where A, B are arbitrary constants; with a modification of these constants, the equation takes the form

$$6.7 \quad \frac{d^2K}{d\theta^2}+\left(\frac{l+\frac{1}{2}}{a+\theta}+\frac{m+\frac{1}{2}}{b+\theta}+\frac{n+\frac{1}{2}}{c+\theta}\right)\frac{dK}{d\theta}$$

$$+\tfrac{1}{4}\left\{\frac{l(l-1)}{(a+\theta)^2}+\frac{m(m-1)}{(b+\theta)^2}+\frac{n(n-1)}{(c+\theta)^2}-\frac{A'\theta+B'}{(a+\theta)(b+\theta)(c+\theta)}\right\}K=0.$$

To multiply each of the functions $K_r(\theta_r)$ by $a_r^{\lambda}b_r^{\mu}c_r^{\nu}$, where λ, μ, ν are constants independent of r, is to multiply the product $\Lambda\mathrm{MN}$ by a constant multiple of $x^{2\lambda}y^{2\mu}z^{2\nu}$. Hence if $K(\theta)$ is a solution of 6.6, $(a+\theta)^{\lambda}(b+\theta)^{\mu}(c+\theta)^{\nu}K(\theta)$ is a solution of the equation derived from 6.6 by replacing l, m, n by $l-2\lambda$, $m-2\mu$, $n-2\nu$. In particular, the solutions of 6.6 are the solutions of the equation

$$6.8 \quad \frac{d^2K}{d\theta^2}+\tfrac{1}{2}\left(\frac{1}{a+\theta}+\frac{1}{b+\theta}+\frac{1}{c+\theta}\right)\frac{dK}{d\theta}=\frac{A\theta+B}{4(a+\theta)(b+\theta)(c+\theta)}K$$

divided by $(a+\theta)^{l/2}(b+\theta)^{m/2}(c+\theta)^{n/2}$. This equation 6.8, which is Lamé's equation, corresponds to zero values of l, m, n, and is therefore the equation satisfied by each of the three factors of any harmonic function that is normal in the confocal variables.

7. If k has one of the values $-a$, $-b$, $-c$, the product $(\theta_1-k)(\theta_2-k)(\theta_3-k)$ is a constant multiple of x^2, y^2, or z^2 as the case may be; for any other value of k this product is seen by 4.3 to have the value $(a+k)(b+k)(c+k)\,\Theta(k)$, that is, to be a multiple of $Su^2/(d+k)-1$ by a factor independent of the variables x, y, z.

A solution ϕ of Laplace's equation which has the form given by

$$7.1 \qquad \phi = Cx^l y^m z^n \prod_p \left(\frac{x^2}{a+k_p}+\frac{y^2}{b+k_p}+\frac{z^2}{c+k_p}-1\right)^{q_p},$$

where C, l, m, n, k_1, q_1, k_2, q_2, ... are constants and the number of factors is finite, is called an *ellipsoidal harmonic*. Since

$$\prod_p \{Su^2/(d+k_p)-1\}^{q_p}$$

is a constant multiple of

$$\prod_p \prod_r (\theta_r - k_p)^{q_p},$$

that is, of $Z(\theta_1)\,Z(\theta_2)\,Z(\theta_3)$, where

$$7.2 \qquad Z(\theta) = \prod_p (\theta-k_p)^{q_p},$$

the function $Z(\theta)$ defined by 7.2 must be a solution of the equation 6.6 in order that the function ϕ given by 7.1 may satisfy Laplace's equation. Since no solution of 6.6 can have an accessible pole or branch point except at one of the critical points $-a$, $-b$, $-c$, and since the constant k_p cannot have either of these critical values, the index q_p must be a positive integer, and therefore the function $Z(\theta)$ must be a polynomial in θ, of which neither $a+\theta$, $b+\theta$, nor $c+\theta$ is a factor. Conversely, any polynomial which satisfies 6.6, resolved into factors, provides an ellipsoidal harmonic that is determinate but for a constant factor.

It is not to be expected that for arbitrary values of the constants involved the equation 6.6 should possess an integral that is a mere polynomial in θ. Let $P_0(z)$ denote generically a power series $p_0+p_1z+p_2z^2+\ldots$ in which p_0 is not zero, and $P_{01}(z)$ a power series in which neither p_0 nor p_1 is zero. Then if in Lamé's equation we substitute $(a+\theta)^\lambda P_0(a+\theta)$ for K, the lowest degree in $(a+\theta)$ that occurs is $\lambda-2$, and the coefficient of $(a+\theta)^{\lambda-2}$ is $p_0\{\lambda(\lambda-1)+\frac{1}{2}\lambda\}$, that is, $p_0\lambda(\lambda-\frac{1}{2})$. Hence one integral of the equation is expansible near $-a$ as $P_0(a+\theta)$ and one integral as $(a+\theta)^{\frac{1}{2}}P_0(a+\theta)$, and any integral which is not a multiple of one of these two is expansible as $P_{01}(a+\theta)^{\frac{1}{2}}$. It follows that the equation 6.6 has an integral A_0 whose form near $-a$ is $(a+\theta)^{-l/2}P_0(a+\theta)$, and an integral A_1 whose form near $-a$ is $(a+\theta)^{-(l-1)/2}P_0(a+\theta)$, and that every integral which is not a multiple of either A_0 or A_1 is of the form $(a+\theta)^{-l/2}P_{01}(a+\theta)^{\frac{1}{2}}$ near $-a$. Hence unless each of the numbers l, m, n has one of the two values 0, 1, *every* solution of 6.6 has either a zero, a pole, or a branch-point at one of the three points $-a$, $-b$, $-c$; in particular, a polynomial solution without one of the factors $a+\theta$, $b+\theta$, $c+\theta$ is then impossible. Thus the values of l, m, n are restricted, not by any analogy with questions in spherical harmonics, but by the nature of the problem proposed in the definition of an ellipsoidal harmonic, and ellipsoidal harmonics fall naturally into four species, since of the values of the three indices l, m, n, all may be zero, two zero and one unity, one zero and two unity, or all unity.

When one of the eight possible sets of values has been chosen for l, m, n, the values of A and B have to be considered. If the degree of the polynomial $Z(\theta)$ is h, the coefficient of the highest power of θ when $Z(\theta)$ is substituted for K in 6.6 is

$$h(h-1)+h(l+m+n+\tfrac{3}{2})$$
$$+\tfrac{1}{4}\{(l+m+n)(l+m+n+3)-2(l+m+n)-A\},$$

and for this to vanish

$$A=4h^2+2h(2l+2m+2n+1)+(l+m+n)(l+m+n+1),$$

that is,

$$A=q(q+1), \tag{7.3}$$

where

$$q=2h+l+m+n, \tag{7.4}$$

that is, where q is *the degree of* ϕ *in the variables* x, y, z.

Thus we are led to regard as the most important case of Lamé's equation the case in which A is of the form $q(q+1)$, where q is a positive integer, and it is of this case that complete solutions have been given, for an arbitrary value of B, by Hermite and Halphen.

But an arbitrary value of B does not permit a polynomial solution of 6.6. If in 6.6, multiplied thoughout by

$$4(a+\theta)(b+\theta)(c+\theta),$$

we substitute
$$K(\theta)=\sum_0^h f_s\theta^s,$$

where the coefficients $f_0, f_1, \ldots, f_h$ are at our disposal, the coefficient of θ^{h+1} vanishes on account of 7.3 and 7.4, the coefficients of θ^h, θ^{h-1} are of the forms

$$-2(2q-1)f_{h-1}+(b_h-B)f_h,$$

$$-4(2q-3)f_{h-2}+(b_{h-1}-B)f_{h-1}+c_{h-1}f_h,$$

the coefficient of θ^s for every value of s from $h-3$ to 1 is of the form

$$-2(h-s)(2q-2h+2s+1)f_{s-1}+(b_s-B)f_s+c_s f_{s+1}+d_s f_{s+2},$$

and the term independent of θ is of the form

$$(b_0-B)f_0+c_0f_1+d_0f_2,$$

where
$$b_0, b_1, \ldots, b_h, c_0, c_1, \ldots, c_{h-1}, d_0, d_1, \ldots, d_{h-2}$$

do not involve B and are polynomials in a, b, c, with coefficients depending on l, m, n, h. If we take f_h to be unity and equate to zero the coefficients of $\theta^h, \theta^{h-1}, \ldots, \theta^2, \theta$, we obtain values of the coefficients $f_{h-1}, f_{h-2}, \ldots, f_1, f_0$ in succession as polynomials in a, b, c, B, and the coefficient of B^s in f_{h-s} is

$$(-1)^s/\{2.4\ldots2(h-s)\}\{(2q-1)(2q-3)\ldots(2q-2h+2s+1)\}.$$

Equating finally to zero the term independent of θ, we have an equation of degree $h+1$ in B, in which the coefficient of B^{h+1} is a constant different from zero and the coefficients of the other powers of B are polynomials in a, b, c. There is a polynomial solution of 6.6 of degree h if and only if B satisfies this equation; to each root of the equation in B corresponds one and only one polynomial, and different values of B lead necessarily to different values of f_{h-1} and therefore to different polynomials in θ and to different ellipsoidal harmonics. Lamé, Liouville, and others have shewn that if a, b, c are real and distinct the roots of the equation in B are also real and distinct, from which it follows that the number of distinct harmonics to which the equation leads is exactly $h+1$; hence in this case the number of distinct ellipsoidal harmonics of degree q is

$$\{\tfrac{1}{2}q+1\}+3\{\tfrac{1}{2}(q-2)+1\}$$

if q is even and is

$$3\{\tfrac{1}{2}(q-1)+1\}+\{\tfrac{1}{2}(q-3)+1\}$$

if q is odd, that is, is always $2q+1$, and these are real functions of x, y, z. For complex values of a, b, c, it is possible, but exceptional, for the equation in B to have equal roots, and therefore for the number of ellipsoidal harmonics of degree q to fall below $2q+1$.

8. Niven has shewn that ellipsoidal harmonics are derivable from harmonics homogeneous in x, y, z by operators constructed from the operator $a(\partial^2/\partial x^2)+b(\partial^2/\partial y^2)+c(\partial^2/\partial z^2)$, that is, $Sd\partial^2/\partial u^2$, which is denoted by D^2. It is interesting to express this operator in terms of the confocal variables, although there is no obvious use for the transformation, since it is essentially to functions of x, y, z that the operators have been applied. To derive an expression for $D^2\phi$ from 3.4, all that is necessary is a determination of the two sums

$$\sum_t d^t\gamma^{st}\gamma^{rt},\ \sum_t d^t\gamma^{st}\gamma_s^{\ rt}$$

for all values of r and s, where d^1, d^2, d^3 are used as alternative symbols for a, b, c.

Writing the identity

$$Su^2/d_q=1$$

in the form
$$Sd_pu^2/d_pd_q=1,$$

that is,
$$\sum_t(d^t+\theta_p)\gamma^{pt}\gamma^{qt}=1/4H_{(p)}H_{(q)},$$

we have from 1.2

$$8.1 \quad \sum_t d^t(\gamma^{pt})^2=(1/4H_{(p)}^{\ 2})-\theta_p=(1-4\theta_pH_{(p)}^{\ 2})/4H_{(p)}^{\ 2},$$

and from 1.4

$$8.2 \qquad \sum_t d^t\gamma^{pt}\gamma^{qt}=1/4H_{(p)}H_{(q)},\quad p\neq q.$$

By differentiation of 8.1

$$8.3 \qquad \sum_t d^t\gamma^{pt}\gamma_p^{\ pt}=-(H_{(p)p}+2H_{(p)}^{\ 3})/4H_{(p)}^{\ 3}.$$

Also from 2.1

$$H_{(r)}\sum_t d^t\gamma^{st}\gamma_s^{rt} = H_{(s)}\sum_t d^t\gamma^{st}\gamma_r^{st} + H_{(s)r}\sum_t d^t(\gamma^{st})^2 - H_{(r)s}\sum_t d^t\gamma^{st}\gamma^{rt},$$

that is,

$$H_{(r)}H_{(s)}\sum_t d^t\gamma^{st}\gamma_s^{rt} = \tfrac{1}{2}\partial\{H_{(s)}^2\sum_t d^t(\gamma^{st})^2\}/\partial\theta_r - H_{(s)}H_{(r)s}\sum_t d^t\gamma^{st}\gamma^{rt},$$

$$= -\tfrac{1}{2}\partial(\theta_s H_{(s)}^2)/\partial\theta_r - H_{(s)}H_{(r)s}\sum_t d^t\gamma^{st}\gamma^{rt},$$

from 8.1, and therefore

8.4 $$\sum_t d^t\gamma^{st}\gamma_s^{rt} = -(4\theta_s H_{(r)}H_{(s)}H_{(s)r} + H_{(r)s})/4H_{(r)}, \quad r \neq s;$$

if s is different from r, the variable θ_r enters into $H_{(s)}^2$ only in the factor $1/(\theta_r - \theta_s)$, hence

$$H_{(s)r}/H_{(s)} = 1/2(\theta_r - \theta_s) = -H_{(r)s}/H_{(r)},$$

and 8.4 can be written

8.5 $$\sum_t d^t\gamma^{st}\gamma_s^{rt} = -(1 - 4\theta_s H_{(s)}^2)H_{(r)s}/4H_{(r)}, \quad r \neq s.$$

We have now from 3.4

$$D^2\phi = \sum_r\left[\frac{1-4\theta_r H_{(r)}^2}{4H_{(r)}^3}\,\frac{\partial}{\partial\theta_r}\left(\frac{\phi_r}{H_r}\right) - \frac{H_{(r)r}+2H_{(r)}^3}{4H_{(r)}^4}\,\frac{\phi_r}{H_{(r)}}\right.$$

$$\left. + \sum_{s\neq r}{}' \left\{\frac{1}{4H_{(r)}H_{(s)}^2}\,\frac{\partial}{\partial\theta_s}\left(\frac{\phi_r}{H_{(r)}}\right) - \frac{(1-4\theta_s H_{(s)}^2)H_{(r)s}}{4H_{(r)}^2H_{(s)}^2}\,\frac{\phi_r}{H_{(r)}}\right\}\right]$$

$$= \sum_r\left[\sum_s \frac{1}{4H_{(s)}^2}\left\{\frac{1}{H_{(r)}}\,\frac{\partial}{\partial\theta_s}\left(\frac{\phi_r}{H_{(r)}}\right) + \frac{\phi_r}{H_{(r)}}\,\frac{\partial}{\partial\theta_s}\left(\frac{1}{H_{(r)}}\right)\right\}\right.$$

$$\left. - \frac{\phi_r}{2H_{(r)}^2} - \frac{\theta_r\phi_{rr}}{H_{(r)}^2} + \sum_s \frac{\theta_s H_{(r)s}\phi_r}{H_{(r)}^3}\right]$$

$$= \sum_r\sum_s \frac{1}{2H_{(s)}^2}\,\frac{\partial}{\partial\theta_s}\left(\frac{\phi_r}{2H_{(r)}^2}\right) - \sum_r \frac{\phi_r}{2H_{(r)}^2}$$

$$- \sum_r \frac{\theta_r\phi_{rr}}{H_{(r)}^2} + \sum_r\left\{\frac{\phi_r}{H_{(r)}^3}\sum_s \theta_s H_{(r)s}\right\}.$$

If we write

8.6 $$E\phi \equiv \sum_r \frac{1}{2H_{(r)}^2}\,\frac{\partial\phi}{\partial\theta_r},$$

the first group of terms here is $E'\phi$ and the second is $E\phi$. Because $\Delta_r H_{(r)}$ is a homogeneous function of θ_1, θ_2, θ_3 of the first degree,

$$\sum_s \theta_s \frac{\partial(\Delta_r H_{(r)})}{\partial\theta_s} = \Delta_r H_{(r)},$$

and because Δ_r is a function of θ_r only,

$$\sum_s \theta_s \frac{\partial\Delta_r}{\partial\theta_s} = \theta_r \frac{\partial\Delta_r}{\partial\theta_r};$$

hence

$$8.7 \qquad \Delta_r \sum_s \theta_s H_{(r)s} = \left(\Delta_r - \theta_r \frac{\partial\Delta_r}{\partial\theta_r}\right) H_{(r)},$$

$$\sum_r \left(\frac{\theta_r\phi_{rr}}{H_{(r)}{}^2} - \frac{\phi_r}{H_{(r)}{}^3}\sum_s \theta_s H_{(r)s}\right) = \sum_r \left\{\frac{\theta_r}{\Delta_r H_{(r)}{}^2}\left(\Delta_r \frac{\partial\phi_r}{\partial\theta_r} + \phi_r \frac{\partial\Delta_r}{\partial\theta_r}\right) - \frac{\phi_r}{H_{(r)}{}^2}\right\},$$

and so finally

$$8.8 \qquad D^2\phi = E^2\phi + E\phi - \sum_r \frac{\theta_r}{\Delta_r H_{(r)}{}^2} \frac{\partial}{\partial\theta_r}\left(\Delta_r \frac{\partial\phi}{\partial\theta_r}\right),$$

that is,

$$8.9 \quad \tfrac{1}{4}(\theta_2 - \theta_3)(\theta_3 - \theta_1)(\theta_1 - \theta_2)\{D^2\phi - E^2\phi - E\phi\}$$

$$= \theta_1(\theta_2 - \theta_3)\Delta_1 \frac{\partial}{\partial\theta_1}\left(\Delta_1 \frac{\partial\phi}{\partial\theta_1}\right) + \theta_2(\theta_3 - \theta_1)\Delta_2 \frac{\partial}{\partial\theta_2}\left(\Delta_2 \frac{\partial\phi}{\partial\theta_2}\right)$$

$$+ \theta_3(\theta_1 - \theta_2)\Delta_3 \frac{\partial}{\partial\theta_3}\left(\Delta_3 \frac{\partial\phi}{\partial\theta_3}\right).$$

From 3.3 and 1.1,

$$x\phi_x + y\phi_y + z\phi_z = \sum_r (x\gamma^{r1} + y\gamma^{r2} + z\gamma^{r3})\,\phi_r / H_{(r)}$$

$$= \sum_r (xx_r + yy_r + zz_r)\,\phi_r / H_{(r)}{}^2$$

for any triply orthogonal transformation; if the variables are confocal, 4.5 implies

$$Suu_r = Su^2/2d_r = \tfrac{1}{2}.$$

Hence for confocal variables

$$E\phi \equiv x\phi_x + y\phi_y + z\phi_z,$$

and ϕ is a homogeneous function of degree n in the rectangular coordinates x, y, z if and only if $E\phi = n\phi$.

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The genesis of Lamé's equation

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